3-Feb

Monday, February 3, 2025 1:57 PM

announcement: the Gradescope invites will be sent sometime before Wed's class

then, on Wed, I'll discuss homework

continuing in § 1.2:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 + t - 4r \\ t \\ 2 \quad + r \\ 3 \quad + r \\ r \end{bmatrix} \xrightarrow[\text{augmented matrix}]{\text{write as an}} E = \begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \end{array}$$

14-Feb: corrected

← infinitely many solutions

called <u>reduced row echelon form</u> (rref)

properties:

- if a row has non-zero entries then first non-zero entry is 1 ← "pivot"
- if a column contains a pivot then all other entries in that column are 0 ← zero
- if a row contains a pivot then each row above contain a pivot to the left

rref(A)

↑ original matrix

ex. $\mathrm{rref}(B) = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ ← inconsistent ⇒ <u>no</u> solutions

process of putting a given matrix in rref called <u>Gauss-Jordan elimination</u>

<u>Section 1.3</u> <u>Matrix Algebra</u>

ex. given the following reduced row echelon forms of augmented matrices

$\mathrm{rref}(A) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \end{array}\right]$ $\quad \mathrm{rref}(B) = \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 1 & 2 & 0 & 1 \end{array}\right]$ $\quad \mathrm{rref}(C) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \end{array}\right]$

$$\text{rref}(A) = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{rref}(B) = \left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{rref}(C) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

infinitely many sol'ns

how many solutions does each matrix have?

inconsistent
(no solutions)

$x_1 = 1 - 2x_2$
$x_3 = 2$
let $x_2 = t$

$$\begin{bmatrix} 1-2t \\ t \\ 2 \end{bmatrix} \begin{array}{l} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{array}$$

no free variables

one solution :

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

<u>rank</u> of a matrix − # of leading 1's in rref(A)
denoted as rank(A)

ex. find rank $\begin{bmatrix} 1 & 2 & 3 \\ ④ & 5 & 6 \\ ⑦ & 8 & 9 \end{bmatrix} \begin{array}{l} \text{I} \\ \\ \text{III} \end{array} = \boxed{2}$

need to first row-reduce given matrix

$$\begin{array}{ccc} -4 & -8 & -12 \\ 4 & 5 & 6 \\ \hline 0 & -3 & -6 \end{array} \xrightarrow{\div -3} \quad 0 \quad 1 \quad 2$$

$$\downarrow$$

$$\text{same} \circlearrowleft \begin{bmatrix} 1 & ② & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{ccc} -7 & -14 & -21 \\ 7 & 8 & 9 \\ \hline 0 & -6 & -12 \end{array} \xrightarrow{\div -6} \quad 0 \quad 1 \quad 2$$

$$\downarrow$$

$$\text{II} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ 0 & -2 & -4 \\ \hline 1 & 0 & -1 \end{array}$$

recall : $n \times m$ matrix where n is # rows
m is # columns

A system has exactly one solution when :

$$\text{rank}(A) = m = n$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

coefficient matrix

i.e., $n \times n$ diagonal matrix

with 1's along diagonal

called <u>identity</u> matrix

Sum of Matrices

must be the same size

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}+\begin{bmatrix}e&f\\g&h\end{bmatrix}=\begin{bmatrix}a+e&b+f\\c+g&d+h\end{bmatrix}$$

2x2 + 2x2 ↑ result is also a 2x2 matrix

Scalar Multiplication with a Matrix

given k is a constant then $k\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}ka&kb\\kc&kd\end{bmatrix}$

ex. simplify $3\begin{bmatrix}2&1\\-1&3\end{bmatrix}=\begin{bmatrix}6&3\\-3&9\end{bmatrix}$

Dot Product of Vectors

given $\vec{v}$ has n components $v_1, \dots, v_n$ ← each vector has same # of components

$\vec{w}$ " $w_1, \dots, w_n$

$\vec{v}, \vec{w}$ may be either row or column vectors

then dot product $\vec{v}\cdot\vec{w}$ is the scalar s.t.

$$\boxed{\vec{v}\cdot\vec{w} := v_1w_1 + v_2w_2 + \dots + v_nw_n}$$

ex. $\begin{bmatrix}1_{v_1}\\2_{v_2}\\3_{v_3}\end{bmatrix}\begin{bmatrix}\underset{w_1}{3}&\underset{w_2}{1}&\underset{w_3}{2}\end{bmatrix} = \overset{v_1w_1}{1(3)} + \overset{v_2w_2}{2(1)} + \overset{v_3w_3}{3(2)}$

$= 3 + 2 + 6 = \boxed{11}$

solution is a value not a matrix

Product of a Matrix with a Row Vector

given A = $n \times m$ matrix with row vectors : $\vec{w}_1, \dots, \vec{w}_n$

$\vec{x}$ is a row vector in $\mathbb{R}^n$

then $A\vec{x} = \begin{bmatrix}\vec{w}_1 & \cdots\cdots \\ \vec{w}_2 & \cdots\cdots \\ \vdots & \\ \vec{w}_n & \cdots\cdots\end{bmatrix}\vec{x} = \begin{bmatrix}\vec{w}_1\cdot\vec{x}\\ \vec{w}_2\cdot\vec{x}\\ \vdots \\ \vec{w}_n\cdot\vec{x}\end{bmatrix}$ ← dot product

ex. $\begin{bmatrix}1&2&3\\1&0&-1\end{bmatrix}\begin{bmatrix}3\\1\\2\end{bmatrix} = \begin{bmatrix}1(3)+2(1)+3(2)\\1(3)+0(1)-1(2)\end{bmatrix} = \begin{bmatrix}3+2+6\\3+0-2\end{bmatrix} = \begin{bmatrix}11\\1\end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 0(1) - 1(2) \end{bmatrix} = \begin{bmatrix} 3 + 0 - 2 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$2 \times \underbrace{3 \quad 3}_{\text{works when inner values are same}} \times 1$

solution: 2×1

ex $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ← is undefined due to a mismatch

$2 \times \textcircled{3} \quad \textcircled{2} \times 1$

not same

ex. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1(x_1) + 0(x_2) + 0(x_3) \\ 0(x_1) + 1(x_2) + 0(x_3) \\ 0(x_1) + 0(x_2) + 1(x_3) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$3 \times 3 \quad 3 \times 1$ → result will be 3×1

recall: this is an identity matrix

re-visit $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{matrix} \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow x_3 \end{matrix}$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

column vectors

each component is a scalar

$$= 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+2+6 \\ 3+0-2 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$$

Same as before ☺